

In the name of Allah, the Beneficent, the Merciful.

A note on a separating system of rational invariants for finite dimensional generic algebras

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Abstract. The paper deals with a construction of a separating system of rational invariants for finite dimensional generic algebras. In the process of dealing an approach to a rough classification of finite dimensional algebras is offered by attaching them some quadratic forms.

INTRODUCTION

In [1] we have offered an approach to classification problem of finite dimensional algebras with respect to basis changes. It has been shown that if one has a special map with some properties then he is able to classify, to list canonical representations, all algebras who's set of structural constants, with respect to a fixed basis, do not nullify some polynomial. In this case he is also able to provide a separating system of rational invariants for those algebras. It was successfully applied in [2] to get a complete classification of all 2-dimensional algebras over algebraically closed fields.

Unfortunately, so far we have no example of such a map in 3-dimensional case. Therefore in the current paper we deal with a weaker problem, namely with a construction of separating system of rational invariants for finite dimensional generic algebras. The theoretical existence of such system of invariants is known [3]. By generic algebras we mean the set of all algebras who's system of structural constants does not nullify a fixed nonzero polynomial in structural variables, over the basic field F . In process of dealing with the problem we show a way for a rough classification of finite dimensional algebras by attaching them some quadratic forms.

The next section contains the main results.

MAIN RESULTS

Further whenever $A = (a_{ij}) \in \text{Mat}(p \times q, F)$, $B \in \text{Mat}(p' \times q', F)$ we use $A \otimes B$ for the matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{2q}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{pmatrix}, \text{ where } F \text{ — is a field of characteristic not 2.}$$

Let us consider any m -dimensional algebra \mathbf{A} with multiplication \cdot given by a bilinear map $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$. If $e = (e_1, e_2, \dots, e_m)$ is a basis for \mathbf{A} then one can represent the bilinear map by a matrix

$$A_e = (A_{e_{jk}}^i)_{i,j,k=1,2,\dots,m} \in \text{Mat}(m \times m^2; F),$$

where $e_j \cdot e_k = e_1 A_{e_{jk}}^1 + e_2 A_{e_{jk}}^2 + \dots + e_m A_{e_{jk}}^m$, $j, k = 1, 2, \dots, m$, such that

$$\mathbf{u} \cdot \mathbf{v} = e A_e (u \otimes v)$$

for any $\mathbf{u} = eu, \mathbf{v} = ev$, where $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m)$ are column vectors. So the algebra \mathbf{A} (binary operation, bilinear map, tensor) is presented by the matrix $A_e \in \text{Mat}(m \times m^2; F)$ -the matrix of structure constants (MSC) of \mathbf{A} with respect to the basis e .

If $e' = (e'_1, e'_2, \dots, e'_m)$ is also a basis for \mathbf{A} , $g \in GL(m, F)$, $e'g = e$ then it is well known that

$$A_{e'} = g A_e (g^{-1})^{\otimes 2}$$

is valid. Further a basis e is fixed and therefore instead of A_e we use A , we do not make difference between \mathbf{A} and its matrix A . Let $X = (X_{jk}^i)_{i,j,k=1,2,\dots,m}$ stand for a variable matrix and $Tr_1(X), Tr_2(X)$ stand for the row vectors

$$\left(\sum_{i=1}^m X_{i1}^i, \sum_{i=1}^m X_{i2}^i, \dots, \sum_{i=1}^m X_{im}^i \right), \left(\sum_{i=1}^m X_{1i}^i, \sum_{i=1}^m X_{2i}^i, \dots, \sum_{i=1}^m X_{mi}^i \right),$$

respectively.

We use τ for the representation of $GL(m, F)$ on the $n = m^3$ dimensional vector space $V = \text{Mat}(m \times m^2; F)$ defined by

$$\tau : (g, A) \mapsto B = g A (g^{-1} \otimes g^{-1}).$$

For simplicity instead of " τ -equivalent", " τ -invariant" we use "equivalent" and "invariant".

We represent each MSC A as a row vector with entries from $\text{Mat}(m, F)$ by parting it consequently into elements of $\text{Mat}(m, F)$:

$$A = (A_1, A_2, \dots, A_m), A_1, A_2, \dots, A_m \in \text{Mat}(m, F).$$

If C is a block matrix with blocks from $\text{Mat}(m, F)$ we use notation $C^{\bar{*}}$, where $*$ is the tensor product or transpose operation, to mean that the operation $*$ with C is done "over $\text{Mat}(m, F)$ " (not over F), for example for the above presented matrix A

$$A^{\bar{t}} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} - \text{column vector over } \text{Mat}(m, F),$$

$$A^{\bar{\otimes 2}} = (A_1^2, A_1 A_2, \dots, A_1 A_m, A_2 A_1, A_2^2, \dots, A_2 A_m, \dots, A_m A_1, A_m A_2, \dots, A_m^2).$$

One can see that the equality $B = g A (g^{-1} \otimes g^{-1})$ can be presented as

$$B = (B_1, B_2, \dots, B_m) = g A (g^{-1})^{\otimes 2} = (g A_1 g^{-1}, g A_2 g^{-1}, \dots, g A_m g^{-1}) (g^{-1} \otimes I),$$

where I stands for $m \times m$ size identity matrix. Moreover for any matrices C and D the equality

$$(C \otimes I) \bar{\otimes} (D \otimes I) = (C \otimes D) \otimes I$$

holds true. Therefore the following equalities hold true.

$$\begin{aligned} (B_1, B_2, \dots, B_m)^{\bar{\otimes k}} &= (g A_1 g^{-1}, g A_2 g^{-1}, \dots, g A_m g^{-1})^{\bar{\otimes k}} ((g^{-1})^{\otimes k} \otimes I), \\ \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}^{\bar{\otimes k}} (B_1, B_2, \dots, B_m)^{\bar{\otimes k}} &= \begin{pmatrix} B_1^2 & B_1 B_2 & \dots & B_1 B_m \\ B_2 B_1 & B_2^2 & \dots & B_2 B_m \\ \vdots & \vdots & \dots & \vdots \\ B_m B_1 & B_m B_2 & \dots & B_m^2 \end{pmatrix}^{\bar{\otimes k}}, \\ \begin{pmatrix} B_1^2 & B_1 B_2 & \dots & B_1 B_m \\ B_2 B_1 & B_2^2 & \dots & B_2 B_m \\ \vdots & \vdots & \dots & \vdots \\ B_m B_1 & B_m B_2 & \dots & B_m^2 \end{pmatrix}^{\bar{\otimes k}} &= \end{aligned}$$

$$((g^t)^{-1})^{\otimes k} \otimes I \left(\begin{pmatrix} gA_1^2g^{-1} & gA_1A_2g^{-1} & \cdots & gA_1A_mg^{-1} \\ gA_2A_1g^{-1} & gA_2^2g^{-1} & \cdots & gA_2A_mg^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ gA_mA_1g^{-1} & gA_mA_2g^{-1} & \cdots & gA_m^2g^{-1} \end{pmatrix} \right)^{\overline{\otimes} k} ((g^{-1})^{\otimes k} \otimes I).$$

Component-wise application of trace to this equality, which is denoted by \tilde{Tr} results in

$$\begin{aligned} & \tilde{Tr} \left(\begin{pmatrix} B_1^2 & B_1B_2 & \cdots & B_1B_m \\ B_2B_1 & B_2^2 & \cdots & B_2B_m \\ \vdots & \vdots & \cdots & \vdots \\ B_mB_1 & B_mB_2 & \cdots & B_m^2 \end{pmatrix} \right)^{\overline{\otimes} k} = \\ & ((g^t)^{-1})^{\otimes k} \tilde{Tr} \left(\begin{pmatrix} gA_1^2g^{-1} & gA_1A_2g^{-1} & \cdots & gA_1A_mg^{-1} \\ gA_2A_1g^{-1} & gA_2^2g^{-1} & \cdots & gA_2A_mg^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ gA_mA_1g^{-1} & gA_mA_2g^{-1} & \cdots & gA_m^2g^{-1} \end{pmatrix} \right)^{\overline{\otimes} k} (g^{-1})^{\otimes k} = \\ & ((g^{-1})^{\otimes k})^t \tilde{Tr} \left(\begin{pmatrix} A_1^2 & A_1A_2 & \cdots & A_1A_m \\ A_2A_1 & A_2^2 & \cdots & A_2A_m \\ \vdots & \vdots & \cdots & \vdots \\ A_mA_1 & A_mA_2 & \cdots & A_m^2 \end{pmatrix} \right)^{\overline{\otimes} k} (g^{-1})^{\otimes k}, \end{aligned}$$

as far as for any matrices C, D and E , where D is a block matrix with blocks from $Mat(m, F)$ and $(C \otimes I)D(E \otimes I)$ has meaning, the equality

$$\tilde{Tr}((C \otimes I)D(E \otimes I)) = C\tilde{Tr}(D)E$$

is valid. One can represent the above obtained matrix equality in the following compact form

$$\tilde{Tr}((B^t B)^{\overline{\otimes} k}) = ((g^{-1})^{\otimes k})^t \tilde{Tr}((A^t A)^{\overline{\otimes} k}) (g^{-1})^{\otimes k}.$$

Note that $\tilde{Tr}((A^t A)^{\overline{\otimes} k})$ is a symmetric matrix. The obtained equality allows formulation of the following theorem.

Theorem 1. *Invariants of the quadratic forms given by the matrix $\tilde{Tr}((X^t X)^{\overline{\otimes} k})$ are invariants of the m -dimensional algebras.*

This result can be used for a rough classification of finite dimensional algebras: Two m -dimensional algebras A, B are rough equivalent if the quadratic forms given by matrices

$$\begin{aligned} \tilde{Tr}(A^t A) &= \begin{pmatrix} Tr(A_1^2) & Tr(A_1A_2) & \cdots & Tr(A_1A_m) \\ Tr(A_2A_1) & Tr(A_2^2) & \cdots & Tr(A_2A_m) \\ \vdots & \vdots & \cdots & \vdots \\ Tr(A_mA_1) & Tr(A_mA_2) & \cdots & Tr(A_m^2) \end{pmatrix}, \\ \tilde{Tr}(B^t B) &= \begin{pmatrix} Tr(B_1^2) & Tr(B_1B_2) & \cdots & Tr(B_1B_m) \\ Tr(B_2B_1) & Tr(B_2^2) & \cdots & Tr(B_2B_m) \\ \vdots & \vdots & \cdots & \vdots \\ Tr(B_mB_1) & Tr(B_mB_2) & \cdots & Tr(B_m^2) \end{pmatrix} \end{aligned}$$

are equivalent.

It is clear that entries of $\tilde{Tr}(X^t X)$ are polynomials in components of X and there exists nonsingular matrix $Q(X^t X)$ with rational entries in X such that the matrix

$$\tilde{Tr}(\overline{X^t X}) = (Q(X^t X)^{-1})^t \tilde{Tr}(X^t X) Q(X^t X)^{-1} = D(X)$$

is a diagonal matrix and $Q(g) = I$ whenever g is a nonsingular diagonal matrix, where $\overline{X} = \tau(Q(X^t X), X)$.

In algebraically closed field F case it means that one can define a nonempty invariant open subset $V_0 \subset V$ such that $\tilde{Tr}(\bar{A}^T \bar{A}) = D(A)$ and $D(A)$ is nonsingular whenever $A \in V_0$.

Theorem 2. Two algebras $A, B \in V_0$ are equivalent(isomorphic) if and only if

$$\bar{B} = \tau(g_0, \bar{A}) \text{ for some } g_0 \in GL(m, F) \text{ for which } g_0^t D(B) g_0 = D(A).$$

Proof. If $B = \tau(g, A)$ then $\bar{B} = \tau(Q(B^T B), B) = \tau(Q(B^T B), \tau(g, A)) =$

$$\tau(Q(B^T B)g, A) = \tau(Q(B^T B)gQ(A^T A)^{-1}, \tau(Q(A^T A), A) = \tau(Q(B^T B)gQ(A^T A)^{-1}, \bar{A}),$$

and for $g_0 = Q(B^T B)gQ(A^T A)^{-1}$ one has

$$\begin{aligned} g_0^t D(B) g_0 &= (Q(B^T B)gQ(A^T A)^{-1})^t D(B) Q(B^T B)gQ(A^T A)^{-1} = \\ &= (Q(A^T A)^{-1})^t (g^t (Q(B^T B)^t D(B) Q(B^T B))g) Q(A^T A)^{-1} = \\ &= (Q(A^T A)^{-1})^t (g^t (\tilde{Tr}(B^T B))g) Q(A^T A)^{-1} = (Q(A^T A)^{-1})^t \tilde{Tr}(A^T A) Q(A^T A)^{-1} = D(A). \end{aligned}$$

Visa versa if $\bar{B} = \tau(g_0, \bar{A})$ for some g_0 for which $g_0^t D(B) g_0 = D(A)$ then for $g = Q(B^T B)^{-1} g_0 Q(A^T A)$ one has

$$\begin{aligned} \tau(g, A) &= \tau(Q(B^T B)^{-1} g_0 Q(A^T A), A) = \tau(Q(B^T B)^{-1} g_0, \tau(Q(A^T A), A) = \\ &= \tau(Q(B^T B)^{-1} g_0, \bar{A}) = \tau(Q(B^T B)^{-1}, \tau(g_0, \bar{A})) = \tau(Q(B^T B)^{-1}, \bar{B}) = B. \end{aligned}$$

Assume that there exists matrix $P(X)$, with rational entries with respect to components of X , such that $P(\bar{A})$ is nonsingular for any $A \in V_0$ and the equality

$$P(\tau(g, \bar{A})) = P(\bar{A})g^{-1} \text{ holds true whenever } g^t D(\tau(g, A))g = D(A). \quad (1)$$

Theorem 3. For $A, B \in V_0$ there exists $g_0 \in GL(m, F)$ such that $g_0^t D(B) g_0 = D(A)$ and $\bar{B} = \tau(g_0, \bar{A})$ if and only if

$$\tau(P(\bar{B}), \bar{B}) = \tau(P(\bar{A}), \bar{A}), \quad (P(\bar{B})^{-1})^t D(B) P(\bar{B})^{-1} = (P(\bar{A})^{-1})^t D(A) P(\bar{A})^{-1}.$$

Proof. If $\bar{B} = \tau(g_0, \bar{A})$ and $g_0^t D(B) g_0 = D(A)$ then

$$\tau(P(\bar{B}), \bar{B}) = \tau(P(\tau(g_0, \bar{A})), \tau(g_0, \bar{A})) = \tau(P(\bar{A})g_0^{-1}, \tau(g_0, \bar{A})) = \tau(P(\bar{A}), \bar{A})$$

and $(P(\bar{B})^{-1})^t D(B) P(\bar{B})^{-1} = ((P(\bar{A})g_0^{-1})^{-1})^t D(B) (P(\bar{A})g_0^{-1})^{-1} =$

$$((P(\bar{A})^{-1})^t g_0^t D(B) g_0 P(\bar{A})^{-1} = (P(\bar{A})^{-1})^t D(A) P(\bar{A})^{-1}.$$

Visa versa, if equalities

$$\tau(P(\bar{B}), \bar{B}) = \tau(P(\bar{A}), \bar{A}), \quad (P(\bar{B})^{-1})^t D(B) P(\bar{B})^{-1} = (P(\bar{A})^{-1})^t D(A) P(\bar{A})^{-1}$$

are valid then for $g_0 = P(\bar{B})^{-1} P(\bar{A})$ one has $g_0^t D(B) g_0 = D(A)$ and

$$\tau(g_0, \bar{A}) = \tau(P(\bar{B})^{-1} P(\bar{A}), \bar{A}) = \tau(P(\bar{B})^{-1}, \tau(P(\bar{A}), \bar{A})) = \tau(P(\bar{B})^{-1}, \tau(P(\bar{B}), \bar{B})) = \bar{B}.$$

So Theorems 2 and 3 imply that the system of entries of matrices

$$\tau(P(\bar{X}), \bar{X}), \quad (P(\bar{X})^{-1})^t \tilde{Tr}(\bar{X}^T \bar{X}) P(\bar{X})^{-1}$$

is a separating system of rational invariants for algebras from V_0 .

The above presented results show importance of construction of matrix $P(X)$ with properties (1). Further we discuss a construction of such matrix by the use of rows $r(\bar{A})$ for which the equality

$$r(\tau(g, \bar{A})) = r(\bar{A})g^{-1}$$

is valid, whenever $g^t D(\tau(g, A))g = D(A)$. To construct such rows one can use the following approach.

Assume that the equalities

$$\bar{B} = g\bar{A}(g^{-1})^{\otimes 2}, \quad \tilde{C} = gCg^t$$

are true, where $C^t = C$ and C is a nonsingular matrix. In this case

$$\tilde{C}^{\otimes 2} = g^{\otimes 2} C^{\otimes 2} (g^{\otimes 2})^t, \quad \bar{B}\tilde{C}^{\otimes 2} = g\bar{A}C^{\otimes 2} (g^{\otimes 2})^t, \quad \tilde{C}^{\otimes 2}\bar{B}^t = g^{\otimes 2} C^{\otimes 2} \bar{A}^t g^t$$

and

$$\bar{B}\tilde{C}^{\otimes 2}\bar{B}^t = g\bar{A}C^{\otimes 2}\bar{A}^t g^t.$$

On induction it is easy to see that for any natural k the equality

$$\bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}} \tilde{C}^{\otimes 2^k} (\bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}})^t = g\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}} C^{\otimes 2^k} (\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}})^t g^t$$

holds true. Therefore due to the equalities

$$\begin{aligned} & \bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}} \tilde{C}^{\otimes 2^k} (\bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}})^t \tilde{C}^{-1} = \\ & g\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}} C^{\otimes 2^k} (\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}})^t C^{-1} g^{-1}, \\ & \bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}} \tilde{C}^{\otimes 2^k} (\bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}})^t \tilde{C}^{-1} \bar{B} = \\ & g\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}} C^{\otimes 2^k} (\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}})^t C^{-1} \bar{A} (g^{-1})^{\otimes 2} \end{aligned}$$

one has

$$\begin{aligned} & Tr_i(\bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}} \tilde{C}^{\otimes 2^k} (\bar{B}^{\otimes 2^0} \bar{B}^{\otimes 2^1} \dots \bar{B}^{\otimes 2^{k-1}})^t \tilde{C}^{-1} \bar{B}) = \\ & Tr_i(\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}} C^{\otimes 2^k} (\bar{A}^{\otimes 2^0} \bar{A}^{\otimes 2^1} \dots \bar{A}^{\otimes 2^{k-1}})^t C^{-1} \bar{A}) g^{-1}, \quad i = 1, 2. \end{aligned}$$

The last equality shows that in our algebra case one can try to construct the needed matrix $P(X)$ by the use of rows

$$Tr_i(X^{\otimes 2^0} X^{\otimes 2^1} \dots X^{\otimes 2^{k-1}} ((\tilde{Tr}(X^t X))^{-1})^{\otimes 2^k} (X^{\otimes 2^0} X^{\otimes 2^1} \dots X^{\otimes 2^{k-1}})^t \tilde{Tr}(X^t X) X),$$

where $i = 1, 2, k = 0, 1, 2, \dots$

What is left unjustified here is that one should justify existence, in general, of a linear independent system consisting of m such rows.

Remark. After a rough classification one can classify further each case of the rough classification with respect to the corresponding stabilizer.

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